

# A note on super Catalan numbers

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May 22, 2012

## Abstract

We show that the super Catalan numbers are special values of the Krawtchouk polynomials by deriving an expression for the super Catalan numbers in terms of a signed set.

The super Catalan numbers,

$$S(m, n) := \frac{(2m)!(2n)!}{m!n!(m+n)!},$$

as designated by Gessel in [9, Eq. (28)] form, hierarchically speaking, special cases of super ballot numbers (cf. [9, pp. 180, 189]). Historically, Gessel points out that these numbers had been observed as early as 1874 and studied by E. Catalan [6]; Aguiar and Hsiao in [1] provide a more detailed account of earlier appearances (cf. [5], [8], [7] and Riordan [14, Chapter 3, Exercise 9, p. 120]). Surprisingly, the innocent looking numbers,  $S(m, n)$ , seem to have been immune against a combinatorial interpretation for all values of  $(m, n)$  over the last century despite limited success stories for particular values (see problem 66(a) in Stanley's bijective open problems compendium [16]). The following references highlight the success cases. In [9], Gessel notes that for  $S(1, n)/2$  we obtain the Catalan number  $C_n$ ; whereas for the case when  $m = 0$ , we yield middle binomial coefficients,  $\binom{2n}{n}$ . In [10], Gessel and Xin provide a combinatorial interpretation in terms of Dyck paths when  $m = 2$  or 3. An alternative combinatorial interpretation for the case  $m = 2$  was provided by Schaeffer in [15] using a method that was introduced in the interpretation to formulas of Tutte for planar maps. A more topologically flavored yet still combinatorial interpretation for the  $m = 2$  case is also available by Pippenger and Schleich in [13]; they count cubic trees on  $n$  interior vertices (or the number of hexagonal trees with  $n$  nodes). In 2005, Callan in [4] provided an elegant

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combinatorial interpretation of the recurrence  $S(m, n)/2 = \sum_{k \geq 0} 2^{n-m-2k} \binom{n-m}{2k} S(m, k)/2$  for the case when  $m = 2$  showing that it enumerates the aligned cubic trees by number of vertices that are neither a leaf nor adjacent to a leaf.

In this note we establish the following expression for super Catalan numbers:

$$S(m, n) = (-1)^m \sum_{P \in \mathcal{P}_{m+n}} (-1)^{h_{2m}(P)}, \quad (1)$$

where the sum is over the set  $\mathcal{P}_{m+n}$  of all lattice paths from  $(0, 0)$  to  $(m+n, m+n)$  consisting of unit steps to the right and up, and  $h_{2m}(P)$  denotes the height of  $P = (P_0, P_1, \dots, P_{2(m+n)}) \in \mathcal{P}_{m+n}$  after the  $2m^{\text{th}}$  step, i.e., the  $y$ -coordinate of  $P_{2m}$ . Although this is an interpretation of  $S(m, n)$  in terms of a signed set only, the right-hand side of (1) is a special value of the Krawtchouk polynomial defined as follows:

$$K_j^d(x) = \sum_{h=0}^j (-1)^h \binom{x}{h} \binom{d-x}{j-h}.$$

Then (1) is equivalent to

$$K_{m+n}^{2(m+n)}(2m) = (-1)^m S(m, n). \quad (2)$$

To see the equivalence, observe that each  $P \in \mathcal{P}_{m+n}$  has exactly  $m+n$  up-steps and that the number of  $P \in \mathcal{P}_{m+n}$  with  $h_{2m}(P) = h$  is therefore equal to  $\binom{2m}{h} \binom{2n}{m+n-h}$ .

Krawtchouk polynomials  $K_j^d(x)$  appear as the coefficients of the so-called MacWilliams identities (cf. [12, Chap. 5, §2]), and also as the eigenvalues of the distance- $j$  graph of the  $d$ -cube (cf. [3, Chap. 3, §2]). The identity shows that  $\{(-1)^m S(m, n) \mid m, n \geq 0, m+n=N\}$  coincides with the set of non-zero eigenvalues of the distance- $N$  graph of the  $2N$ -cube, which is known as the orthogonality graph and has been studied in connection with pseudo-telepathy in quantum information theory (cf. [11]). Finally, (2) follows immediately from the identity of von Szily (cf. [9, Eq. (29)]):

$$\begin{aligned} S(m, n) &= \sum_{k \in \mathbb{Z}} (-1)^k \binom{2m}{m+k} \binom{2n}{n-k} \\ &= (-1)^m \sum_{h=0}^{m+n} (-1)^h \binom{2m}{h} \binom{2n}{m+n-h} \\ &= (-1)^m K_{m+n}^{2(m+n)}(2m). \end{aligned}$$

We note that (2), as well as the identity of von Szily, is just a restatement of (a special case of) Kummer's evaluation of well-poised  ${}_2F_1(-1)$  series.

To obtain a proper interpretation as the size of a set of certain paths, we need to find an injection from the set

$$\{P \in \mathcal{P}_{m+n} \mid h_{2m}(P) \not\equiv m \pmod{2}\}$$

to

$$\{P \in \mathcal{P}_{m+n} \mid h_{2m}(P) \equiv m \pmod{2}\},$$

and a description of the complement of the image. This is known for the case  $m = 1$  (see [2, Section 5.3]), but it seems to be a difficult problem in general.

## Acknowledgments

Special thanks are due to Ole Warnaar and Richard Askey for valuable comments. E.G. is indebted to M. Sipser and J. Tsitsiklis of MIT and would like to thank I. Gessel of Brandeis University for helpful background comments on this problem in early 2010. H.T. was supported in part by the JSPS Excellent Young Researchers Overseas Visit Program.

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